Lazzeri's Jacobian of oriented compact riemannian manifolds

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Math. classification: Primary 14K30; Secondary 53C99.
Keywords: Jacobians, riemannian manifolds.

Abstract The subject of this paper is a Jacobian, introduced by F. Lazzeri (unpublished), associated to every compact oriented riemannian manifold whose dimension is twice an odd number. We start the investigation of Torelli type problems and Schottky type problems for Lazzeri's Jacobian; in particular we examine the case of tori with flat metrics. Besides we study Lazzeri's Jacobian for Kähler manifolds and its relationship with other Jacobians. Finally we examine Lazzeri's Jacobian of a bundle.

1. Introduction

The subject of this paper is a Jacobian, introduced by F. Lazzeri (unpublished), associated to every compact oriented riemannian manifold whose dimension is twice an odd number; if (M, g) is a compact oriented riemannian manifold of dimension $n = 2m = 2(2k + 1)$, Lazzeri's Jacobian of (M, g) is the following p.p.a.v.:
the torus $H^m(M, \mathbf{R})/(H^m(M, \mathbf{Z})/torsion)$ with the complex structure given by the operator $*$ and the polarization whose imaginary part is $(\alpha, \beta) = -\int \alpha \wedge \beta$, $\alpha, \beta \in H^m(M, \mathbf{Z})$.
(It is really a p.p.a.v., in fact: the operator $* : H^m(M, \mathbf{R}) \to H^m(M, \mathbf{R})$ (defined through the isomorphism of $H^m(M, \mathbf{R})$ with the space of harmonic m-forms) has square -1, so it induces a complex structure on $H^m(M, \mathbf{R})$; besides $\int \cdot \wedge \cdot = \int * \cdot \wedge * \cdot$ and $\int \cdot \wedge *\cdot$ is positive definite.)
One can easily see that the following definition is good:
Definitions 1.1. *Let $k \in \mathbf{N}$ and $m = 2k+1$. Let $\mathcal{R} \subset \{(M, g) \mid M$ oriented compact C^∞ manifold of dimension $2m$, g riemannian metric on $M\}$. Let $\sim$ be the following equivalence relation on $\mathcal{R}$: $(M_1, g_1) \sim (M_2, g_2)$ iff there are an orientation preserving diffeomorphism $f : M_1 \to M_2$ and a C^∞ map $t : M_1 \to \mathbf{R}^+$ such that $(M_1, t\, g_1) \in \mathcal{R}$ and $f^* g_2 = t\, g_1$. Let $\mathcal{A}_h$ be the set of p.p.a.v.'s of dimension h up to isomorphisms. Let*

$$T_{\mathcal{R}} : \mathcal{R}/\sim \;\longrightarrow\; \mathcal{A}_{1/2\, b_m(M)}$$

*be the map associating to the class of (M, g) Lazzeri'Jacobian of (M, g).
Now fix M and a symplectic basis of $H^m(M, \mathbf{Z})/torsion$ with respect $-\int \cdot \wedge \cdot$ and let $\mathcal{R} \subset \{g \mid g$ metric on $M\}$. Then $T_{\mathcal{R}}$ can be lifted to a map*

$$\hat{T}_{\mathcal{R}} : \mathcal{R}/conformal\, equivalence \;\longrightarrow\; \mathcal{H}_{1/2\, b_m(M)},$$

*where $\mathcal{H}_h$ is the h-Siegel upper half space (two metrics on M, g_1 and g_2, are said conformal equivalent iff $g_1 = tg_2$ where t is a C^∞ map: $M \to \mathbf{R}^+$).
We often omit the subscripts $\mathcal{R}$ in $T_{\mathcal{R}}$ and $\hat{T}_{\mathcal{R}}$.*
If we consider the metrics of constant curvature equal to -1 on a C^∞ compact surface, we have that Lazzeri's Jacobian is the usual Jacobian of the correspondent Riemann surface.
The outline of the paper is the following: Section 2 deals with Torelli and Schottky problem for Lazzeri's Jacobian of tori with flat metrics, in Sect. 3 we study Lazzeri's Jacobian for Kähler

manifolds and its relationship with other Jacobians and in Sect. 4 we examine Lazzeri's Jacobian of a bundle.

To state our results on Lazzeri's Jacobian we fix here some notation we will use in all the paper.

Notation 1.2. • $\boldsymbol{\mathcal{A}_h}$, $\boldsymbol{\mathcal{H}_h}$: *defined above; we will often drop the subscripts h.*

• $\boldsymbol{G_{\mathbf{R}}(h,V), G_{\mathbf{C}}(k,W)}$. *$G_{\mathbf{R}}(h,V)$ is the Grassmannian of the real h-subspaces of the real vector space V, $G_{\mathbf{C}}(k,W)$ is the Grassmannian of the complex k-subspaces of the complex vector space W.*

• $\boldsymbol{\mathcal{F}_n}$. *We define $\mathcal{F}_n = \{(\mathbf{R}^n/\Lambda, g) \mid \Lambda$ lattice, g flat metric on $\mathbf{R}^n/\Lambda\}/\sim$, where $\sim$ is the equivalence relation defined in Def. 1.1; here it becomes $(\mathbf{R}^n/\Lambda, g) \sim (\mathbf{R}^n/\Lambda', g')$ iff $\exists$ a orientation preserving map $f : \mathbf{R}^n/\Lambda \to \mathbf{R}^n/\Lambda'$ induced by a linear map $\mathbf{R}^n \to \mathbf{R}^n$ such that $f^*g' = cg$ for some $c \in \mathbf{R}^+$, where the orientation is the standard one of $\mathbf{R}^n$, (in fact if $(\mathbf{R}^n/\Lambda, g)$ and $(\mathbf{R}^n/\Lambda', g')$ are equivalent for $\sim$ through a map φ, then φ is given by an affine map $\mathbf{R}^n \to \mathbf{R}^n$).*

• $\boldsymbol{\mathcal{T}_n}$. *Set $\mathcal{T}_n := \{T \in M(n\times n, \mathbf{R}) \mid T$ lower triangular, $T_{i,i} > 0 \;\forall i$, $det\, T = 1\}$.*

• *Let $m \in \mathbf{N}$ and $n = 2m$. Consider the standard basis of $\mathbf{R}^n$, $\{e_i\}_{i=1,..n}$. We call* **lexicographic ordered basis** *of $\wedge^m\mathbf{R}^n$ the following ordered basis: e_I, $I = (i_1, ..., i_m) \in \mathbf{N}^m$ with $1 \leq i_1 < i_2 < ... < i_m \leq n$, with the multindeces ordered by the lexicographic order. Let $\mathcal{I} := \{I = (1, i_2, .., i_m) \in \mathbf{N}^m \mid 1 < i_2 < .. < i_m \leq n\}$. If $I \in \mathcal{I}$, we choose and fix forever one of the multindeces J such that we obtain (I, J) from $(1, ..., n)$ with an even number of transpositions and we call it $\tilde{I}$. Let $\mathcal{E} = \{\tilde{I} \mid I \in \mathcal{I}\}$. We want to define another ordered basis of $\wedge^m\mathbf{R}^n$: we take first the multindices in $\mathcal{I}$ in the lexicographic order: we call them $I_1, I_2, ...$; then we consider $\tilde{I}_1, \tilde{I}_2, ...$; we call* **the symmetric lexicographic ordered basis** *of $\wedge^m\mathbf{R}^n$ the ordered basis $e_{I_1}, e_{I_2}, ..., e_{\tilde{I}_1}, e_{\tilde{I}_2}, ...$, and we call "the symmetric lexicographic order" the order $I_1, I_2, ..., \tilde{I}_1, \tilde{I}_2, ...$ of the multindeces of $\mathcal{I} \cup \mathcal{E}$. Analogously for $(\mathbf{R}^n)^\vee$ instead of $\mathbf{R}^n$.*

• *Let M be a complex manifold. If $H^q(M, \mathbf{C}) = V \oplus W$, $\pi_{V,W}$ will denote the projection onto V. We will often omit the subscript V, W in $\pi_{V,W}$ when it will be clear.*

• $\boldsymbol{K(M,g)}$, $\boldsymbol{K'(M,g)}$, $\boldsymbol{F^p_q(M)}$, $\boldsymbol{J_q(M)}$. *Let (M, g) be a compact Kähler manifold of complex dimension m; let Ω be the (1,1)-form associated to g.*

$K = K(M, g) := \{[\eta] \in H^m(M, \mathbf{C}) | \, [\eta] = \sum_{r\geq 0,\, r\equiv -m(m+1)/2 \;(mod\, 2)}[\Omega^r\eta_r]$ with η_r primitive form of degree $m - 2r\}$ and

$K' = K'(M, g) := \{[\eta] \in H^m(M, \mathbf{C}) | \, [\eta] = \sum_{r\geq 0,\, r\equiv 1-m(m+1)/2 \;(mod\, 2)}[\Omega^r\eta_r]$ with η_r primitive form of degree $m - 2r\}$.

$F^p_q(M) := \oplus_{a+b=q,\, a\geq p} H^{a,b}(M, \mathbf{C})$ (we will omit the subscript q when no confusion can arise).

If q is an odd number, we define $J_q(M) = \oplus_{a+b=q,\, a-b\equiv 1\; (4)} H^{a,b}(M, \mathbf{C})$.

First we examine the case of flat metrics on tori. Let $n = 2m = 2(2k+1)$ and $N := \frac{1}{2}\dim \wedge^m\mathbf{R}^n$; we study $T : \mathcal{F}_n \to \mathcal{A}_N$ and, choosing a symplectic basis, we study $\hat{T}$ and in particular Im $\hat{T}$ (and then Im T):

Theorem A. i) *Choosen any symplectic basis of $H^m(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z})$ with respect to $-\int \cdot \wedge \cdot$, the map $\hat{T} : \{g \mid g$ flat metric on $\mathbf{R}^n/\mathbf{Z}^n\}/conf.\ equiv. \longrightarrow \mathcal{H}_N$ is injective.*

ii) *Now call $\{dx_1, ..., dx_n\}$ the standard basis of $(\mathbf{R}^n)^\vee$. If we take as a symplectic basis of $H^m(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z})$ with respect to $-\int \cdot \wedge \cdot$, the sym. lex. ordered basis $\{dx_I = dx_{i_1} \wedge ... \wedge dx_{i_m}\}_{I\in\mathcal{I}\cup\mathcal{E}}$, we have that $Im\hat{T} = \{X + iY \in \mathcal{H}_N \mid 1)$ and $2)$ hold$\}$, where:*

1) $Y = EE^t$ where $E_{I,J} = det(T)_{I,J}$, $I, J \in \mathcal{I}$, $\mathcal{I}$ ordered in lex. order., for some T upper triangular matrix $n \times n$ with determinant 1 and $T_{i,i} \geq 0$ (in particular E is upper triangular with positive diagonal elements and the entries of every column of E are the Plücker coordinates of an element of $G_{\mathbf{R}}(2k, \mathbf{R}^{n-1})$ and the same for the entries of every row of E).

2) $X_{IJ} = 0$ if I and J have more than one index in common, where $I, J \in \mathcal{I}$, $\mathcal{I}$ ordered in lex. order.

iii) *Lazzeri's Jacobian of a generic flat oriented torus has not nontrivial automorphisms as p.p.a.v..*

iv) *The map $T : \mathcal{F}_n \longrightarrow \mathcal{A}_N$ is generically locally injective,* **v)** *but not injective.*

In Section 3 we consider Lazzeri's Jacobian for Kähler manifolds (M, g) and we see that it depends only on the complex structure of M and on the cohomology class of the $(1,1)$-form associated to g;

using that, we examine the relationship of Lazzeri's Jacobian with Weil's and Griffiths' Jacobians:
Theorem B. *Let (M,g) be a compact Kähler manifold of complex dimension $m=2k+1$; (in the sequel π is $\pi_{J_m(M),\overline{J_m(M)}}$); we have that, as complex tori,*
- *k^{th} Weil's Jacobian is $H^m(M,\mathbf{R})/(H^m(M,\mathbf{Z})/torsion)$ with the complex structure given by C (or $J_m(M)/\pi(H^m(M,\mathbf{Z}))$ with the complex structure given by i),*
- *k^{th} Griffiths' Jacobian is $H^m(M,\mathbf{R})/(H^m(M,\mathbf{Z})/torsion)$ with the complex structure given by C on the set $\{\eta+\overline{\eta} \mid \eta \in J_m(M)\cap\overline{F^{k+1}(M)}\}$ and $-C$ on $\{\eta+\overline{\eta} \mid \eta\in J_m(M)\cap F^{k+1}(M)\}$ (or $J_m(M)/\pi(H^m(M,\mathbf{Z}))$ with the complex structure $i|_{J_m(M)\cap\overline{F^{k+1}(M)}}\oplus -i|_{J_m(M)\cap F^{k+1}(M)}$),*
- *Lazzeri's Jacobian is $H^m(M,\mathbf{R})/(H^m(M,\mathbf{Z})/torsion)$ with the complex structure $C|_{H^m(M,\mathbf{R})\cap K}\oplus -C|_{H^m(M,\mathbf{R})\cap K'}$ (or $J_m(M)/\pi(H^m(M,\mathbf{Z}))$ with the complex structure $i|_{J_m(M)\cap K}\oplus -i|_{J_m(M)\cap K'}$);*

k^{th} Weil's Jacobian and Lazzeri's Jacobian are p.p.a.v.'s (also if the 1-1 form associated to the metric isn't rational) and the real part of the polarization is the same (equal to $\int \cdot\wedge *\cdot$ on $H^m(M,\mathbf{R})/(H^m(M,\mathbf{Z})/torsion)$).
Besides we consider another class of Ricci-flat metrics (besides the flat ones on tori) Kähler-Einstein metrics on the complex manifolds with trivial canonical bundle and we find another local Torelli theorem for Lazzeri's Jacobian (**Corollary C**), and one could formulate the following conjecture:
Conjecture. *$\hat{T}_{\mathcal{R}}$ is locally injective if $\mathcal{R}$ is a set of Ricci-flat metrics on a manifold M.*
Finally in Section 4 given a bundle $F\to M$, we study the relationship between Lazzeri's Jacobian of F and Lazzeri's Jacobian of M **(Prop. D)**.
We think that one can easily find open problems about Lazzeri's Jacobian, for instance to go on with the study of Schottky and Torelli type problems, to study Prym-Tyurin varieties for Lazzeri's Jacobians and the relationship between Lazzeri's Jacobian and the theory of degeneration of abelian varieties or more precisely to study a possible "object" $T(M,g_0)$ to associate to every compact oriented manifold M of dimension $2(2k+1)$ with a singular metric g_0 (i.e. $(g_0)_P$ is semipositive definite $\forall P\in M$), such that, if g_t are riemannian metrics on M for $t>0$ and $g_t\longrightarrow g_0$ for $t\longrightarrow 0$, then $T(M,g_t)\longrightarrow T(M,g_0)$ in some sense. Observe that if we consider a torus $M=\mathbf{R}^{2(2k+1)}/\Lambda$ with a singular flat metric g_0, one could define $T(M,g_0)$ in the following way: let g be a flat metric on M extending g_0 (i.e. $(g_0)_P=(g)_P$ on $(\ker(g_0)_P)^{\perp_g}$ $\forall P\in M$, or equivalently for one point of M); we define $T(M,g_0)=\mathcal{H}^{2k+1}_g(M,\mathbf{R})/\psi(\mathcal{H}^{2k+1}_g(M,\mathbf{Z}))$ with the complex structure $*_g$, where $\mathcal{H}^{2k+1}_g$ is the set of harmonic $(2k+1)$-forms for g, i.e. invariant $(2k+1)$-forms, ψ is defined pointwise as the projection onto $\wedge^{2k+1}((\ker(g_0)_P)^{\perp_g})^\vee$ $\forall P$ and $*_g$ the operator $*$ for g (one can easily see that this definition doesn't depend on the metric g extending g_0, see Appendix.)

2. The case of tori with flat metrics

2.a. Some lemmas of linear algebra

Lemma 2.1. *Let Ω be a real upper triangular matrix $n\times n$, where $n=2m$. In the lex. ordered basis of $\wedge^m\mathbf{R}^n$, the matrix $\bigwedge^m\Omega$ is upper triangular. Besides, if $\bigwedge^m\Omega$ is equal to $\begin{pmatrix} A & C\\ B & D\end{pmatrix}$ in the sym. lex. ordered basis of $\wedge^m\mathbf{R}^n$, then $B=0$, A is upper triangular and D lower triangular.*
Proof. Left to the reader. □
Lemma 2.2. *Let Ω be a real upper triangular matrix $n\times n$, where $n=2m$.*
Let us fix an ordered basis of $\wedge^m\mathbf{R}^n$ defined in the following way: order in any fixed way the multindeces I in $\mathcal{I}$; call them $K_1,K_2,..,$ then, order the multindices in $\mathcal{E}$ in the following way: $\tilde{K}_1,\tilde{K}_2,..$; consider $e_{K_1},e_{K_2},...,e_{\tilde{K}_1},e_{\tilde{K}_2},...$; (for instance the sym. lex. ordered basis).
In this ordered basis of $\wedge^m\mathbf{R}^n$ let $\bigwedge^m\Omega$ be $\begin{pmatrix} A & C\\ B & D\end{pmatrix}$. We have:
a)$A^tD=(\det\Omega)I$;
b)suppose $\det\Omega\neq 0$; for $I\in\mathcal{I}$ and $J\in\mathcal{E}$

$$(A^{-1}C)_{IJ} = \begin{cases} 0 & if\ \tilde{I}\ and\ J\ have\ more\ than\ one\ index\ in\ common \\ \pm\frac{\Omega_{1,l}}{\Omega_{1,1}} & if\ \tilde{I}\ and\ J\ have\ only\ l\ in\ common \end{cases}$$

Proof. If P and Q are two multindeces, $\Omega_{P,Q}$ will denote the determinant of the minor P,Q of Ω. We will denote the j^{th}-column of Ω by v_j and $v_j - \Omega_{1,j}e_1$ by $\overline{v}_j$.
a) Let $I = (i_1, ..., i_m) \in \mathcal{I}$ and $J = (j_1, ..., j_m) \in \mathcal{E}$. Observe that
$(A^tD)_{I,J}e_1 \wedge ... \wedge e_n = \sum_{S\in\mathcal{I}} \Omega_{S,I}\Omega_{\tilde{S},J}e_1 \wedge ... \wedge e_n =$
$= (\sum_{S\in\mathcal{I}} \Omega_{S,I}e_{s_1} \wedge .... \wedge e_{s_m}) \wedge (\sum_{T\in\mathcal{E}} \Omega_{T,J}e_{t_1} \wedge ... \wedge e_{t_m}) =$
$= (\sum_{S\in\mathcal{I}\cup\mathcal{E}} \Omega_{S,I}e_{s_1} \wedge .... \wedge e_{s_m}) \wedge (\sum_{T\in\mathcal{I}\cup\mathcal{E}} \Omega_{T,J}e_{t_1} \wedge ... \wedge e_{t_m}) =$
$= v_{i_1} \wedge ... \wedge v_{i_m} \wedge v_{j_1} \wedge ... \wedge v_{j_m} = \begin{cases} 0 & \text{if } J \neq \tilde{I} \\ (\det\Omega)\, e_1 \wedge ... \wedge e_n & \text{if } J = \tilde{I} \end{cases}$
where $S = (s_1, ..., s_m)$, $T = (t_1, ..., t_m)$, and the last but one equality holds because $\Omega_{S,I} = 0$ for $S \in \mathcal{E}$ and $I \in \mathcal{I}$.
b) Let $I, J \in \mathcal{E}$. We remark that, since neither I nor J contain 1, I and J have some index in common. Let $I = (i_1, ..., i_m)$ and $J = (j_1, ..., j_m)$ with $l = i_r = j_s$. We have:

$$\begin{aligned} & (\det\Omega)(A^{-1}C)_{IJ}e_1 \wedge ... \wedge e_n \overset{a)}{=} (D^tC)_{IJ}e_1 \wedge ... \wedge e_n = (\sum_{K\in\mathcal{I}} \Omega_{\tilde{K},I}\Omega_{K,J})e_1 \wedge ... \wedge e_n = \\ = \ & \overline{v}_{i_1} \wedge ... \wedge \overline{v}_{i_m} \wedge (v_{j_1} \wedge ... \wedge v_{j_m} - \overline{v}_{j_1} \wedge ... \wedge \overline{v}_{j_m}) = \overline{v}_{i_1} \wedge ... \wedge \overline{v}_{i_m} \wedge v_{j_1} \wedge ... \wedge v_{j_m} = \\ = \ & \overline{v}_{i_1} \wedge ... \wedge \overline{v}_{i_m} \wedge v_{j_1} \wedge ... \wedge v_{j_{s-1}} \wedge \Omega_{1,l}e_1 \wedge v_{j_{s+1}} \wedge ... \wedge v_{j_m} = \\ = \ & \overline{v}_{i_1} \wedge ... \wedge \overline{v}_{i_m} \wedge \overline{v}_{j_1} \wedge ... \wedge \overline{v}_{j_{s-1}} \wedge \Omega_{1,l}e_1 \wedge \overline{v}_{j_{s+1}} \wedge ... \wedge \overline{v}_{j_m} = \\ = \ & \begin{cases} 0 & if\, I\, and\, J\, have\, more\, than\, one\, index\, in\, common \\ \epsilon\,\frac{\Omega_{1,l}}{\Omega_{1,1}}\ \det\Omega\, e_1 \wedge ... \wedge e_n & if\, I\, and\, J\, have\, only\, l\, in\, common \end{cases} \end{aligned}$$

where ϵ is the sign of the permutation taking $(i_1, ..., i_m, j_1, ..., j_{s-1}, 1, j_{s+1}, ..., j_m)$ in $(1, ..., n)$. □
Lemma 2.3. *Let $m \in \mathbf{N}$ and $n = 2m$. Consider the set of positive scalar products on R^n up to conformal equivalence. The map, defined on this set, associating to the class of a positive definite scalar product its operator $*$ on $\wedge^m(\mathbf{R}^n)^\vee$, is injective.*
Proof. We observe that, given a scalar product on $\mathbf{R}^n$, two elements of $\mathbf{R}^n$, v and w, are perpendicular iff there exists a m-subspace of $\mathbf{R}^n$ perpendicular to w and containing v; thus v and w are perpendicular iff $\exists\ \alpha \in \wedge^{m-1}\mathbf{R}^n$, α simple (i.e. of the kind: $v_1 \wedge ... \wedge v_{m-1}$) such that $w \wedge *(\alpha \wedge v) = 0$ and $\alpha \wedge v \neq 0$. Thus $*$ determines the conformal structure. □

2.b. Proof of Theorem A

Remark 2.4. *The set $\{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g$ flat metric on $\mathbf{R}^n/\mathbf{Z}^n\}$/conf. equivalence is in bijection with the set P_n of symmetric positive definite matrices $n \times n$ with determinant 1, thus with $\mathcal{T}_n$. The set $\mathcal{F}_n = \{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g$ flat metric on $\mathbf{R}^n/\mathbf{Z}^n\}/\sim\ = P_n/SL(n,\mathbf{Z})$ (where $A \in SL(n,\mathbf{Z})$ acts on P_n by $P \mapsto A^tPA$) and we endow it with the quotient topology induced by the set above.*

Proposition 2.5. *Let $n := 2m = 2(2k+1)$ and $N := \frac{1}{2}\begin{pmatrix} n \\ m \end{pmatrix}$.*
Let $L \in GL(n,\mathbf{R})$; let $\wedge^{2k+1}(L^{-1})^t = \begin{pmatrix} A(L) & C(L) \\ B(L) & D(L) \end{pmatrix}$ in the sym. lex. order; set $\begin{pmatrix} X(L) \\ Y(L) \end{pmatrix} = \begin{pmatrix} A(L) & -B(L) \\ B(L) & A(L) \end{pmatrix}^{-1} \begin{pmatrix} C(L) \\ D(L) \end{pmatrix}$; define $Z(L) = X(L) + iY(L) \in M(N \times N, \mathbf{C})$.
Let $\det L > 0$. Call $\{dx_1, ..., dx_n\}$ the standard basis of $(\mathbf{R}^n)^\vee$. Choose the sym. lex. ordered basis $\{dx_I = dx_{i_1} \wedge ... \wedge dx_{i_{2k+1}}\}_{I\in\mathcal{I}\cup\mathcal{E}}$ as symplectic basis of $H^m(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z})$ for $-\int \cdot \wedge \cdot$. We have that $Z(L) = \hat{T}(\mathbf{R}^n/\mathbf{Z}^n, L^tL)$ (the orientation of $\mathbf{R}^n/\mathbf{Z}^n$ is given by the standard one of $\mathbf{R}^n$).
Proof. Let b_i be the columns of L^{-1}, they are an orthonormal basis of $\mathbf{R}^n$ for the metric L^tL.

The matrix representing the ordered basis $\{dx_I\}_{I\in\mathcal{I}\cup\mathcal{E}}$ in function of the sym. lex. ordered basis of $H^m(\mathbf{R}^n/\mathbf{Z}^n,\mathbf{R})$, $\{b_I^\vee\}_{I\in\mathcal{I}\cup\mathcal{E}}$ is $\wedge^{2k+1}(L^{-1})^t = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$. Then the matrix expressing $\{dx_I, *dx_I\}_{I\in\mathcal{I}}$ in function of the sym. lex. ordered basis $\{b_I^\vee\}_{I\in\mathcal{I}\cup\mathcal{E}}$ is $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. Thus the matrix expressing $\{dx\}_{I\in\mathcal{E}}$ in function of $\{dx_I, *dx_I\}_{I\in\mathcal{I}}$ is: $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}^{-1}\begin{pmatrix} C \\ D \end{pmatrix}$. □

Proof of Theorem A. In proving ii), iii) and v) we use Prop. 2.5 and its notation.
i) The map $\hat{T}$ is injective by Lemma 2.3 and because the harmonic forms on tori are the translations-invariant forms.
ii) Observe that, if $L\in\mathcal{T}_n$, then $\wedge^{2k+1}(L^{-1})^t = \begin{pmatrix} A(L) & C(L) \\ 0 & (A(L)^t)^{-1} \end{pmatrix}$ by Lemmas 2.1 and 2.2.
Then $X(L) = A(L)^{-1}C(L)$ and Lemma 2.2 implies the claim for X .
Besides $Y(L) = A^{-1}(L)(A^{-1}(L))^t$, where $A(L)_{I,J} = \det((L^{-1})^t)_{I,J}$; then we have $Y(L) = A^{-1}(L)(A^{-1}(L))^t = A(L^{-1})(A(L^{-1}))^t$ where $A(L^{-1})_{I,J} = \det(L^t)_{I,J}$. Taking $E = A(L^{-1})$ and $T = L^t$ we conclude.
iii) Let G be the modular group. If $F_\sigma := \{x\in\mathcal{H}_N \mid \sigma(x) = x\}$ for $\sigma\in G^* := G - \{Id\}$, we have to prove that $\mathcal{T}_n - Z^{-1}(\cup_{\sigma\in G^*}F_\sigma)$ is a open dense subset of $\mathcal{T}_n$. The openness follows from the fact that G acts properly and discontinously on $\mathcal{H}$ (see [L-B] p. 218). To prove the density, it is sufficient (by Baire's theorem) to prove that $\forall\sigma\in G^*$, the set $\mathcal{T}_n - Z^{-1}(F_\sigma)$ is a open dense subset of $\mathcal{T}_n$; since $Z^{-1}(F_\sigma)$ is defined by polynomial equations, we have only to prove that $\mathcal{T}_n - Z^{-1}(F_\sigma)\neq\emptyset$ i.e. that $Z(\mathcal{T}_n)\subset F_\sigma$ implies $\sigma = Identity$.
Let $\sigma\in G$ be the map $Z\mapsto (MZ+N)(PZ+Q)^{-1}$. Obviously $Z\in F_\sigma$ iff $ZPZ+ZQ = MZ+N$. Let $L\in\mathcal{T}_n$; for every $c\in\mathbf{R}^+$, let $\tilde{L}(c)$ be the matrix obtained from L by multiplying $L_{1,1}$ by c^{-4k-1} and the other entries of L by c; we have $Z(\tilde{L}(c)) = c^{4k+2}Z(L)$; since $Z(\tilde{L}(c))\in F_\sigma$ $\forall c\in\mathbf{R}^+$, we have $c^{8k+4}Z(L)PZ(L) + c^{4k+2}(Z(L)Q - MZ(L)) - N = 0$ $\forall c\in\mathbf{R}^+$, $\forall L\in\mathcal{T}_n$; thus:
1) $N = 0$ 2) $Z(L)PZ(L) = 0\ \forall\ L\in\mathcal{T}_n$ 3) $Z(L)Q - MZ(L) = 0\ \forall\ L\in\mathcal{T}_n$
Hence: $Q = (M^t)^{-1}$ by 1) and $\sigma\in G$; $P = 0$, $M = Q$ by 2) and 3), taking $L = I$. Thus M is orthogonal. From 3), we have: $Y(L)M = MY(L)$ and this implies M diagonal (take L in such way that $Y(L)$ diagonal with the elements of the diagonal different one from another). Being M orthogonal and diagonal, M must be diagonal with only ±1 on the diagonal; still from $Y(L) = MY(L)M^{-1}$, taking L in such way that $Y(L)$ not diagonal, we obtain $M = I$.
iv) It follows right away from i) and iii), since G acts properly and discontinously on $\mathcal{H}$.
v) Let us consider two diagonal matrices $\in\mathcal{T}_n$, one inverse of the other: F and F^{-1}.
We have: $C(F) = B(F) = C(F^{-1}) = B(F^{-1}) = 0$ and $A(F) = A(F^{-1})^{-1}$, $D(F) = D(F^{-1})^{-1}$. Thus $Y(F^{-1}) = Y(F)^{-1}$ and $X(F^{-1}) = X(F) = 0$. Thus $Z(F) = -Z(F^{-1})^{-1}$; so the Z's differ by a modular map. Thus Lazzeri's Jacobians of $(\mathbf{R}^n/\mathbf{Z}^n, F^2)$ and of $(\mathbf{R}^n/\mathbf{Z}^n, (F^{-1})^2)$ are isomorphic p.p.a.v.'s. But we can choose F in such way that $\nexists A\in SL(n,\mathbf{Z})$ s.t. $A^tF^2A = (F^{-1})^2$. □

3. The case of Kähler manifolds

We recall some facts on the operators $*$, C and L (see for instance [Ch], [Wei], [G-H], [Wel], [Gre]).

Let M be a complex manifold of complex dimension m; Weil's operator C is defined on the forms of bedegree (a,b) by $C\eta = i^{a-b}\eta$ and then is extended by linearity to arbitrary forms; observe that C takes real forms to real forms. If M is compact and of Kählerian type, then, $\forall q\in\mathbf{N}$ odd, the operator C defines a complex structure on $H^q(M,\mathbf{C})$ (and thus on $H^q(M,\mathbf{R})$) depending only on the complex structure of M; the i-eigenspace of C is $J_q(M) = \oplus_{a+b=q, a-b\equiv 1\ (4)}H^{a,b}(M,\mathbf{C})$.

Now suppose that (M,g) is a hermitian manifold and let Ω be the $(1,1)$-real form associated to g; let L be the operator on the set of forms defined by $L\eta = \Omega\wedge\eta$ and Λ the adjoint operator of L. A form η such that $\Lambda\eta = 0$ is said primitive. Every form ω of degree q can be written uniquely in the form $\omega = \sum_{r\geq(q-m)^+}L^r\omega_r$, where ω_r is a primitive form of degree $q-2r$. If g is of Kähler then L defines an operator on $H^q(M,\mathbf{C})$; a class $[\omega]\in H^q(M,\mathbf{C})$ is said primitive if $[\Lambda\omega] = 0$ and

a class $[\omega] \in H^q(M, \mathbf{C})$ can be written uniquely in the form $[\omega] = \sum_{r \geq (q-m)^+} L^r[\omega_r]$, where $[\omega_r]$ is a primitive class of degree $q - 2r$.

Warning 3.1. Here the operator $*$, defined as usual on real forms, is extended to complex forms by $\mathbf{C}$-linearity as in [Ch] and [Wei].

Lemma 3.2. *Let (M, g) be a hermitian manifold of complex dimension m. Let η be a m-form on M. If we write $\eta = \sum_{r \geq 0} L^r \eta_r$ with η_r primitive $(m-2r)$-form, we have that:*

$$*\eta = \sum_{r \geq 0} (-1)^{\frac{m^2+m}{2}+r} L^r C \eta_r, \qquad C\eta = \sum_{r \geq 0} L^r C \eta_r.$$

Then we observe that we can decompose the space of the m-forms in two parts:
$\{\eta = \sum_{r \geq 0,\, r\, even} L^r \eta_r$ with η_r primitve $(m-2r)$-form$\}$ and
$\{\eta = \sum_{r \geq 0,\, r\, odd} L^r \eta_r$ with η_r primitve $(m-2r)$-form$\}$;
on the first part $* = (-1)^{\frac{m^2+m}{2}} C$, on the second one $* = (-1)^{\frac{m^2+m}{2}+1} C$.

Proof. In [Ch] p. 26 or [Wel] the following formula is proved: *if ω is a primitive p-form on M and $r \leq m - p$ ($m = \dim_{\mathbf{C}} M$), then* $*L^r\omega = (-1)^{\frac{p(p+1)}{2}} \frac{r!}{(m-p-r)!} L^{m-p-r} C\omega$.
Applying it, with $p = m - 2r$, to each term $L^r \eta_r$ of the sum $\eta = \sum_{r \geq 0} L^r \eta_r$, we obtain:

$$*\eta = * \sum_{r \geq 0} L^r \eta_r = \sum_{r \geq 0} * L^r \eta_r = \sum_{r \geq 0} (-1)^{\frac{m^2+m}{2}+r} L^r C \eta_r.$$

Since $CL = LC$, we have:

$$C\eta = \textstyle\sum_{r \geq 0} CL^r \eta_r = \sum_{r \geq 0} L^r C \eta_r. \qquad \square$$

Corollary 3.3. *Let (M, g) be a compact Kähler manifold of complex dimension m. Consider the operators $*$ and C on $H^m(M, \mathbf{C})$. We have that $* = C$ on K and $* = -C$ on K' (see Not. 1.2).*

Let (M, g) be a compact Kähler manifold of complex dimension m. We recall the definitions of Weil's and Griffiths' Jacobians (see for instance [Ch], [Wei], [G-H], [Gre]).
• Suppose that the 1-1 real form Ω associated to the metric is rational; let $p \in \mathbf{N}$ with $p \leq m - 1$; p^{th} *Weil's Jacobian* is an abelian variety so defined:
the torus $H^{2p+1}(M, \mathbf{R})/(H^{2p+1}(M, \mathbf{Z})/torsion)$ with the complex structure given by C and the polarization whose real part is $\mathcal{R}(\alpha, \beta) = \int \alpha \wedge *\beta$.
• Let $p \in \mathbf{N}$, $p \leq m - 1$; p^{th} *Griffiths' Jacobian* is the following complex torus:
the torus $H^{2p+1}(M, \mathbf{C})/(F^{p+1}(M) + H^{2p+1}(M, \mathbf{Z})) = \overline{F^{n-p+1}(M)}/\pi(H^{2p+1}(M, \mathbf{Z}))$ with the complex structure given by i.

Proof of Theorem B. We recall that $H^m(M, \mathbf{C}) = K \oplus K'$ and that, by Corollary 3.3, $* = -C$ on K and $* = C$ on K'. To prove the Theorem we have only to observe that $H^m(M, \mathbf{R}) = (H^m(M, \mathbf{R}) \cap K) \oplus (H^m(M, \mathbf{R}) \cap K')$ and $J_m = (J_m \cap K) \oplus (J_m \cap K')$. $\square$

Thus k^{th}-Griffiths', k^{th}-Weil's, Lazzeri's Jacobians of a Käher manifold (M, g) of dimension $m = 2k + 1$ are the same real tori with a different complex structure and the "change" of the complex structure depends on the complex structure of M for Griffiths' J.- Weil's J. and on the class of the $(1, 1)$-form associated to g for Lazzeri's J.- Weil's J..

	$H^m(M, \mathbf{R})/(H^m(M, \mathbf{Z})/torsion)$	$J_m(M)/\pi(H^m(M, \mathbf{Z}))$
k^{th} G.J.	$C\vert_{\{\eta+\overline{\eta}\vert\ \eta \in J_m(M) \cap \overline{F^{k+1}(M)}\}} \oplus -C\vert_{\{\eta+\overline{\eta}\vert\ \eta \in J_m(M) \cap F^{k+1}(M)\}}$	$i\vert_{J_m(M) \cap \overline{F^{k+1}(M)}} \oplus -i\vert_{J_m(M) \cap F^{k+1}(M)}$
k^{th} W. J.	C	i
L. J.	$C\vert_{H^m(M, \mathbf{R}) \cap K} \oplus -C\vert_{H^m(M, \mathbf{R}) \cap K'}$	$i\vert_{J_m(M) \cap K} \oplus -i\vert_{J_m(M) \cap K'}$

Definition 3.4. *Let (M, g) be a compact Kähler manfold of complex dimension $m = 2k + 1$. Set* $J'_m(M) := (K(M, g) \cap J_m(M)) \oplus \overline{(K'(M, g) \cap J_m(M))}$.

Corollary 3.5. *Let (M,g) be a compact Kähler manfold of complex dimension $m = 2k+1$; we have $T(M,g) = J'_m(M)/\pi_{J'_m(M),\overline{J'_m(M)}}(H^m(M,\mathbf{Z}))$ with the complex structure given by i and the imaginary part of the polarization $(\alpha,\beta) = -\int_M(\alpha+\overline{\alpha})\wedge(\beta+\overline{\beta})$.*

Remark 3.6. Let (M,g) be a compact Kähler manifold of complex dimension $m = 2k+1$; seeing $T(M,g)$ as $J'_m(M)/\pi(H^m(M,\mathbf{Z}))$, we can define a Abel map for k-cycles:
let Z_0 be a k-cycle in M and set $B(Z_0) := \{Z \text{ } k\text{-}cycle\ homologous\ to\ Z_0\}$; let $\{\psi_1,....\psi_l\}$ be a basis of $J'_m(M)$: Abel's map $\mu : B(Z_0) \longrightarrow T(M)$ is so defined: if $Z \in B(Z_0)$ and C is a $(2k+1)$-chain with $\partial C = Z - Z_0$,

$$\mu(Z) := (\int_C \psi_1, ..., \int_C \psi_l).$$

One can easily see that the definition is good (in an analogous way as in [Li] p.131) and by the same calculation in [Gri] p.826, one can see that if $\{Z_\lambda\}_{\lambda\in B}$ is a family of effective k-subvarieties with B not singular and $Z_0 = Z_{\lambda_0}$, the map $\mu(\lambda) = \mu(Z_\lambda)$ is not holomorphic in general: let C_λ be a chain such that $\partial C_\lambda = Z_\lambda - Z_0$; we have that $\int_{C_\lambda}\psi_i$ is 0 unless ψ_i is of kind $(k+1,k)$ or $(k,k+1)$ and that $\int_{C_\lambda}\psi_i$ is holomorphic in the first case and antiholomorphic in the second case.

Notation 3.7. *Let M be a complex compact manifold of dimension m; consider a smooth deformation of the complex structure $\mathcal{M} \to \Delta$, (Δ polycylinder $\ni 0$); we call M_t the fibre over t and let ϕ be a C^∞ trivialization: $\mathcal{M} \to M \times \Delta$ (possibly restricting Δ); ϕ induces diffeomorphism $\phi_t : M_t \longrightarrow M$. Let $\rho : T_0(\Delta) \longrightarrow H^1(\Theta)$ be the Kodaira-Spencer map, where $T_0(\Delta)$ is the holomorphic tangent space to Δ in 0 and $\Theta = \theta(T^{10}(M))$. Suppose that M is of Kählerian type (and then also M_t by Theorem 15 in [K-S]).*

We recall the definition of Griffiths' and Weil's period maps.

• The Griffiths period map $\mathcal{G}_q : \Delta \longrightarrow G_{\mathbf{C}}(f_q^q, H^q(M,\mathbf{C})) \times ... \times G_{\mathbf{C}}(f_q^{q-\nu}, H^q(M,\mathbf{C}))$ (possibly restricting Δ) is the map:

$$t \mapsto (\mathcal{G}_q^q(t), ..., \mathcal{G}_q^{q-\nu}(t)),$$

where $\mathcal{G}_q^p(t) = (\phi_t^{-1})^* F_q^p(M_t)$, $f_q^p = \dim F_q^p(M)$ and $\nu = [\frac{q-1}{2}]$.

• If q is an odd positive integer, the Weil period map $\mathcal{W}_q : \Delta \longrightarrow G_{\mathbf{C}}(b_q/2, H^q(M,\mathbf{C}))$ (possibly restricting Δ) is the map

$$t \mapsto \mathcal{W}_q(t) = (\phi_t^{-1})^* J_q(M_t).$$

Let $\phi(t)$ be a smoothly varying harmonic $(q-r,r)$ form on M_t. Call $\phi = \phi(0)$. We have (see [Gri] p. 812 or [Gre] p. 33):

$\frac{\partial\phi(t)}{\partial t}|_{t=0} = \rho(\frac{\partial}{\partial t})\cdot\phi$, which is of type $(q-r-1, r+1)$, (1)

$\frac{\partial\phi(t)}{\partial \overline{t}}|_{t=0} = \overline{\rho(\frac{\partial}{\partial t})}\cdot\phi$, which is of type $(q-r+1, r-1)$, (2)

where $\cdot$ is the contraction.

Thus we have that, while $\mathcal{G}_q$ is holomorphic and $Im\,(d\mathcal{G}_q)(0)$ is in $\oplus_{r=0...\nu} Hom(H^{q-r,r}, H^{q-r-1,r+1})$, the map $\mathcal{W}_q$ is not holomorphic; precisely, if $\phi \in F_q^{q-r}(M)$

$$(\frac{\partial}{\partial t}\mathcal{G}_q^{q-r})(0)(\phi) = \pi_{\overline{F_q^{r+1}(M)},F_q^{q-r}(M)}(\rho(\frac{\partial}{\partial t})\cdot\phi) = \rho(\frac{\partial}{\partial t})\cdot\phi,$$

$$(\frac{\partial}{\partial \overline{t}}\mathcal{G}_q^{q-r})(0)(\phi) = \pi_{\overline{F_q^{r+1}(M)},F_q^{q-r}(M)}(\overline{\rho(\frac{\partial}{\partial t})}\cdot\phi) = 0;$$

while, if q is odd and $\phi \in J_q(M)$

$$(\frac{\partial}{\partial t}\mathcal{W}_q)(0)(\phi) = \pi_{\overline{J_q(M)},J_q(M)}(\rho(\frac{\partial}{\partial t})\cdot\phi) = \rho(\frac{\partial}{\partial t})\cdot\phi,$$

$$(\frac{\partial}{\partial \overline{t}}\mathcal{W}_q)(0)(\phi) = \pi_{\overline{J_q(M)},J_q(M)}(\overline{\rho(\frac{\partial}{\partial t})}\cdot\phi) = \overline{\rho(\frac{\partial}{\partial t})}\cdot\phi.$$

Corollary C. *Let (M,g) be a compact Kähler manifold of odd complex dimension m; let $\Omega \in H^2(M,\mathbf{R})$ be the class of the $(1,1)$-form associated to g; consider a smooth deformation of the complex structure $\mathcal{M} \to \Delta$ for which we use Notation 3.7. Let $\Delta' = \{t \in \Delta \mid \phi_t^*(\Omega) \in H^2(M_t,\mathbf{R})$ can be represented by a Kähler form for $M_t\}$; suppose Δ' be a neighbourhood of 0.*

a) *For $t \in \Delta'$ let g_t be a Kähler metric on M_t whose (1,1)-form is of class $\phi_t^*(\Omega)$; if we fix a symplectic basis of $H^m(M, \mathbf{Z})$, we have that the map $t \mapsto Z(T(M_t, g_t))$ defined from a neighbourhood of 0 in Δ' to $\mathcal{H}$, associating to t the matrix in $\mathcal{H}$ representing $T(M_t, g_t)$ with that choice of the symplectic basis, is holomorphic in 0 iff $\rho(T_0(\Delta)) \cdot (K(M,g) \cap J_m(M) \oplus \overline{(K'(M,g) \cap J_m(M))}) = 0$ (where $\cdot$ is the contraction), (observe that $t \mapsto Z(T(M_t, g_t))$ is the composition of the map $t \mapsto (\phi_t^{-1})^* g_t$ with $\hat{T}$).*
b) *Suppose now that the canonical bundle of M is trivial and ρ is injective. For $t \in \Delta'$ let g_t be a Kähler metric on M_t whose (1,1)-form is of class $\phi_t^*(\Omega)$, for instance the Kähler-Einstein metric whose (1,1)-form is of class $\phi_t^*(\Omega)$ ($\exists!$ by Calabi-Yau's theorem (see [S.P.]); fix a symplectic basis of $H^m(M, \mathbf{Z})$; then the differential in 0 of the map $t \mapsto Z(T(M_t, g_t))$ (defined above) is injective.*
PROOF. We set $\mathcal{W}'_m(t) = (\phi_t^{-1})^* J'_m(M_t)$. By (1) and (2), we have that, if $\phi \in J'_m(M)$, then
$(\frac{\partial}{\partial t}\mathcal{W}'_m)(0)(\phi) = \pi_{\overline{J'_m(M)}, J'_m(M)}(\rho(\frac{\partial}{\partial t}) \cdot \phi) = \rho(\frac{\partial}{\partial t}) \cdot \phi$,
$(\frac{\partial}{\partial \overline{t}}\mathcal{W}'_m)(0)(\phi) = \pi_{\overline{J'_m(M)}, J'_m(M)}(\overline{\rho(\frac{\partial}{\partial t})} \cdot \phi) = \overline{\rho(\frac{\partial}{\partial t})} \cdot \phi$.
See $T(M_t, g_t) = \mathcal{W}'_m(t)/\pi_{\mathcal{W}'_m(t), \overline{\mathcal{W}'_m(t)}}(H^m(M, \mathbf{Z}))$ with the polarization whose imaginary part is $(\alpha, \beta) = -\int_M (\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta})$. Choosen a symplectic (for $-\int \cdot \wedge \cdot$) basis $\{\gamma_i\}$ of $H^m(M, \mathbf{Z})$ and a basis $\{\omega_j(t)\}$ of $\mathcal{W}'_m(t)$, let $\begin{pmatrix} E(t) \\ F(t) \end{pmatrix}$ be the matrix expressing $\omega_j(t)$ in function of γ_i. Then $Z(T(M_t, g_t)) = -\overline{E(t)F(t)^{-1}}$.
Thus $Z(T(M_t, g_t))$ is holomorphic in t iff $\overline{\mathcal{W}'_m(t)} \in G_{\mathbf{C}}(b_m/2, H^m\ (M, \mathbf{C}))$ is holomorphic in t.
If the canonical bundle of M is trivial, the maps $H^1(\Theta) \to Hom(H^{n,0}, H^{n-1,1})$ and $\overline{H^1(\Theta)} \to Hom(H^{0,n}, H^{1,n-1})$ given by the contraction are injective (see [Gri] p. 844). Then, as in [Gri], if also ρ is injective, the maps $(d\mathcal{W}_m)(0)$ and $(d\mathcal{W}'_m)(0)$ are injective and we obtain b). □

4. Lazzeri's Jacobian of a bundle

Definition 4.1. *An element λ of a lattice Λ is said primitive if $\nexists \lambda' \in \Lambda$ such that $\lambda \in \mathbf{Z}\lambda'$.*

Proposition D. *Let (M, g_M) and (N, g_N) be riemannian compact oriented manifolds of dimension $2(2k+1)$, resp. $2(2s)$. Let $p : F \to M$ be a bundle with fibre N and structure group $G \subset Diff(N)$ and suppose g_N to be G-invariant. We consider on F the metric induced by g_M and g_N. Suppose $\exists\ \lambda \in H^{2s}(N, \mathbf{Z})/torsion$ such that $\lambda \neq 0$, $*_N \lambda = \lambda$ and λ G-invariant.*
We define a map $e_\lambda : T(M) \to T(F)$ (we omit the metrics). If $F = M \times N$ we can define e_λ simply as the map induced by the map $H^{2k+1}(M, \mathbf{R}) \to H^{2k+1+2s}(M \times N, \mathbf{R})$ defined by $\eta \mapsto \eta \wedge \lambda$. More generally define $\Lambda \in H^{2s}(F, \mathbf{R})$ in the following way: if U a trivializing open subset of M, $\Lambda|_{p^{-1}(U)} := \pi^* \lambda$, where π is the composition of a C^∞ trivialization $p^{-1}(U) \to U \times N$ with the projection $U \times N \to N$; suppose Λ is an integral form; define $E_\lambda : H^{2k+1}(M, \mathbf{R}) \to H^{2k+1+2s}(F, \mathbf{R})$ by $E_\lambda(\eta) = p^*\eta \wedge \Lambda$; the map E_λ defines a map $e_\lambda : T(M) \to T(F)$.
The map e_λ is holomorphic and, if θ_F and θ_M are the polarizations of $T(F)$ and of $T(M)$, then $e_\lambda^* \theta_F = (\int_N \lambda \wedge * \lambda)\theta_M$. Besides e_λ is injective if one of the following conditions holds: a) $\int_N \lambda \wedge * \lambda = 1$ b) $F = M \times N$ and λ is primitive c) $H^*(N, \mathbf{Z})$ is free and G-invariant.

PROOF. • The fact that $*_N \lambda = \lambda$ implies at once that e_λ is holomorphic and that $e_\lambda^* \theta_F = (\int_N \lambda \wedge * \lambda)\theta_M$:
let $\{U_\alpha\}_\alpha$ be a trivializing covering of M and let ψ_α be a partition of the unity for this covering; let ϕ_α be the partition of the unity for the covering of F $\{p^{-1}(U_\alpha)\}$ defined by $\phi_\alpha(y) = \psi_\alpha(p(y))$; let $\omega_1, \omega_2 \in H^{2k+1}(M, \mathbf{R})$:

$$\int_F E_\lambda(\omega_1) \wedge E_\lambda(\omega_2) = \sum_\alpha \int_{p^{-1}(U_\alpha)} \phi_\alpha E_\lambda(\omega_1) \wedge E_\lambda(\omega_2) =$$

$$\sum_\alpha \int_{U_\alpha \times N} \phi_\alpha \omega_1 \wedge \lambda \wedge \omega_2 \wedge \lambda = \sum_\alpha \int_{U_\alpha} \psi_\alpha \omega_1 \wedge \omega_2 \int_N \lambda \wedge * \lambda = (\int_N \lambda \wedge * \lambda) \int_M \omega_1 \wedge \omega_2$$

• If a) holds then the map e_λ is injective since it is a homomorphism of p.p.a.v.'s and preserves the polarization. It is easy to verify that b) implies e_λ injective. Finally, if c) holds, then e_λ is injective by the theorem of Leray-Hirsch (see [Sp] p.258). $\square$

Remark 4.2. Suppose $F = M \times N$. Observe that $T(M \times N)$ is the product of the abelian subvarieties A_t corresponding to $(H^t(M, \mathbf{R}) \otimes H^{2k+1+2s-t}(N, \mathbf{R})) \oplus (H^{2(2k+1)-t}(M, \mathbf{R}) \otimes H^{2s-2k-1+t}(N, \mathbf{R}))$ for $t \neq 2k+1$, and the abelian subvariety A_{2k+1} corresponding to $H^{2k+1}(M, \mathbf{R}) \otimes H^{2s}(N, \mathbf{R})$. Obviuosly $Im\ e_\lambda \subset A_{2k+1}$.
The abelian subvarieties A_t for $t \neq 2k+1$ are represented in the Siegel upper half space by a matrix Z with real part equal to 0; in fact:
take as a complex basis of $(H^t(M, \mathbf{R}) \otimes H^{2k+1+2s-t}(N, \mathbf{R})) \oplus (H^{2(2k+1)-t}(M, \mathbf{R}) \otimes H^{2s-2k-1+t}(N, \mathbf{R}))$ a basis $\{r_i\}$ of $H^t(M, \mathbf{Z}) \otimes H^{2k+1+2s-t}(N, \mathbf{Z})$; we can complete $\{r_i\}$ to a symplectic basis $\{r_i, s_i\}$ of the lattice taking as $\{s_i\}$ a suitable basis of $H^{2(2k+1)-t}(M, \mathbf{Z}) \otimes H^{2s-2k-1+t}(N, \mathbf{Z})$; since each s_i is a real linear combination of the $*r_i's$, Z is imaginary.

Remark 4.3. If N is Kähler and the 1-1 form Ω associated to the metric is integral, i.e. N is projective, the hypothesis "$\lambda \in H^{2s}(N, \mathbf{Z})/torsion$, $*_N \lambda = \lambda$" is satisfied if $\lambda = [\Omega^s]$.

Appendix

Let $M = \mathbf{R}^{2(2k+1)}/\Lambda$ be a torus and g_0 a singular flat metric. We prove now that the definition of $T(M, g_0)$ we gave at the end of Introduction does not depend on the metric g extending g_0:
let $\{e_1, ..., e_n\}$ be a basis of $\mathbf{R}^n$ with $n = 2(2k+1)$, orthonormal for g with $< e_1, ..., e_r >= (\ker g_0)^{\perp_g}$ and $< e_{r+1}, ..., e_n >= \ker g_0$;
let g' be another metric extending g_0 in the above sense; let $\{v_1, ..., v_n\}$ be a basis of $\mathbf{R}^n$ orthonormal for g' with $< v_1, ..., v_r >= (\ker g_0)^{\perp_{g'}}$ and $< v_{r+1}, ..., v_n >= \ker g_0$;
let $\begin{pmatrix} A & 0 \\ C & E \end{pmatrix}$ be the matrix expressing the basis $\{v_i\}$ in function of the basis $\{e_i\}$;
observe that the matrix A is orthogonal, in fact $g_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ both in the basis $\{e_1, ..., e_r, e_{r+1}, ..., e_n\}$ and in the basis $\{v_1, ..., v_r, v_{r+1}, ..., v_n\}$;
thus $\begin{pmatrix} A & 0 \\ C & E \end{pmatrix}^t \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ C & E \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$; then A is orthogonal;
consider the basis $\{v'_1, ..., v'_n\}$ of $\mathbf{R}^n$ that is expressed by the matrix $\begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix}$ in function of the basis $\{v_1, ..., v_n\}$;
the matrix expressing $\{v'_1, ..., v'_n\}$ in function of $\{e_1, ..., e_n\}$ is $\begin{pmatrix} I & 0 \\ CA^{-1} & E \end{pmatrix}$;
thus $\{v'_1, ..., v'_n\}$ is a basis orthonormal for g' such that $< v'_1, ..., v'_r >= \ker g_0^{\perp_{g'}}$ and $< v'_{r+1}, ..., v'_n >= \ker g_0$;
the map $e_I^\vee \mapsto (v'_I)^\vee$ is an isomorphism between $T(M, g_0)$ built by g and $T(M, g_0)$ built by g'.

Acknowledgements. I thank prof. F. Catanese for suggesting me to study Lazzeri's Jacobian and for many helpful discussions and suggestions on it. I thank also M. Manetti for some corrections to the first part of this work.

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